Biased random walks

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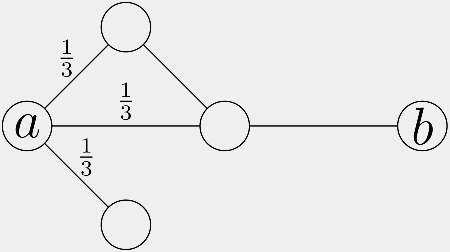
- Let $G = (V, E)$ be a graph, $a \neq b \in V$. A simple random walk is a randomly generated sequence of vertices (v_i) such that $v_1 = a, v_{i+1} \in N(v_i)$ and v_{i+1} is chosen uniformly at random.



Introduction

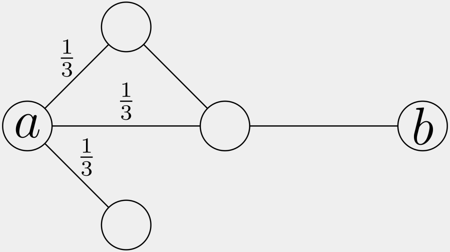
- Let $G = (V, E)$ be a graph, $a \neq b \in V$. A simple random walk is a randomly generated sequence of vertices (v_i) such that $v_1 = a, v_{i+1} \in N(v_i)$ and v_{i+1} is chosen uniformly at random.
- The hitting time of b is the number of steps the walk needs to reach b from a.

Known results

It was shown [Aleliunas et al., 1979, F. Lawler, 1986] that the expected hitting time on any connected undirected graph is of order $\mathcal{O}(n^3)$.
To be precise, the expected hitting time is at most $\frac{4}{27}n^3 - \frac{1}{9}n^2 + \frac{2}{3}n - 1$ [Brightwell and Winkler, 1990].

Biased walk

Given a graph $G = (V, E)$, choose some vertices $F \subseteq V$ and a target $b \in V$. In these 'excited' vertices, the random walker will deterministically take a step along a fixed shortest path to b.



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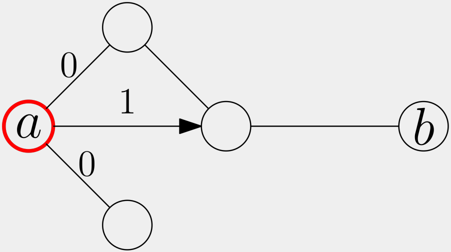

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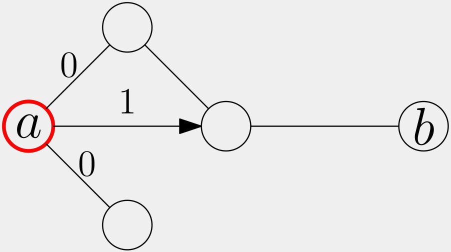

Question

Does the hitting time of b change, and if so, how?

- Can we show the same $\mathcal{O}(n^3)$ bound on the expected hitting time as before?

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- Are there any other natural ‘biases’, which help the random walker?

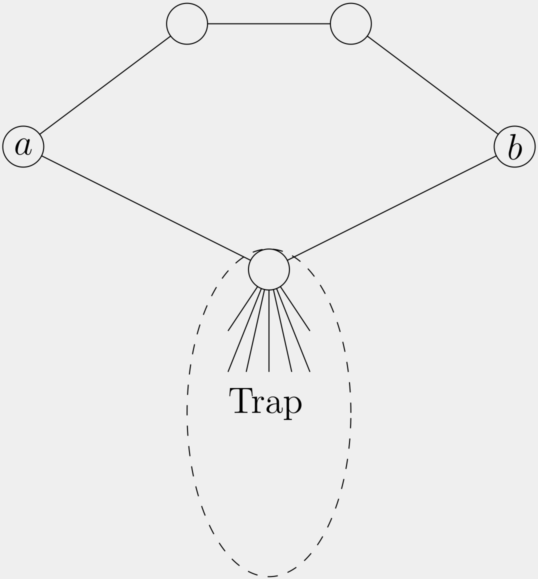
Problems and results

- Can we show the same $\mathcal{O}(n^3)$ bound on the expected hitting time as before? No.
- Can we show any polynomial bound? No.
- Are there any other natural ‘biases’, which help the random walker? ¯_(ツ)_/¯

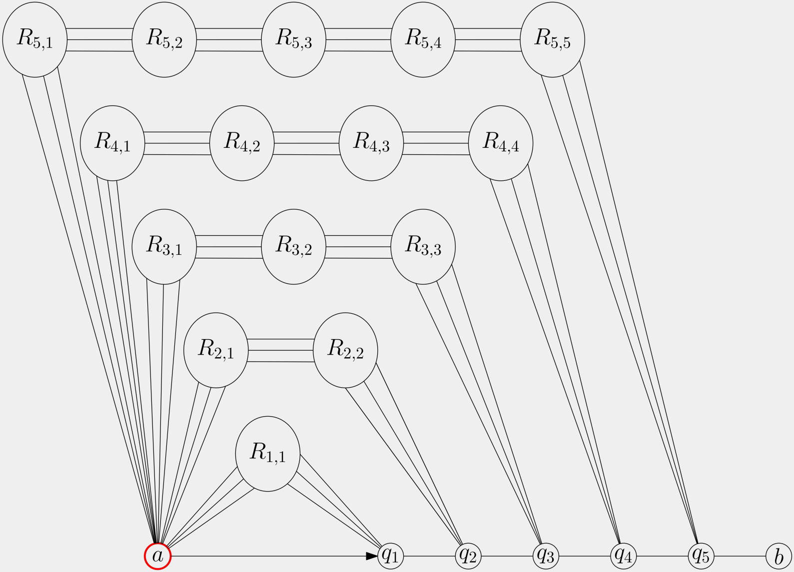
Theorem

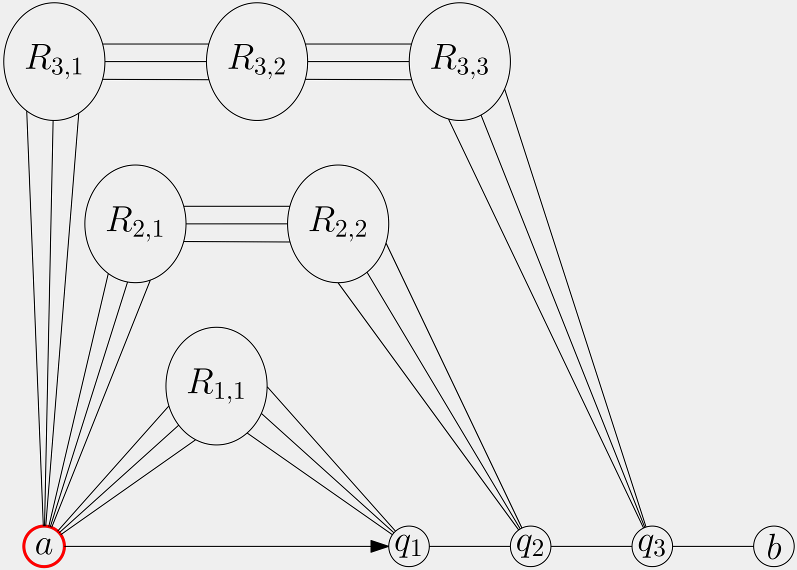
For any $c \in \mathbb{N}$, there exists a graph $G = (V, E), |V| = n$ with vertices $a, b \in V$ such that the expected hitting time of b when starting in a is $\Omega(n^c)$. Moreover, only one excited vertex is required.

Superpolynomiality - general idea



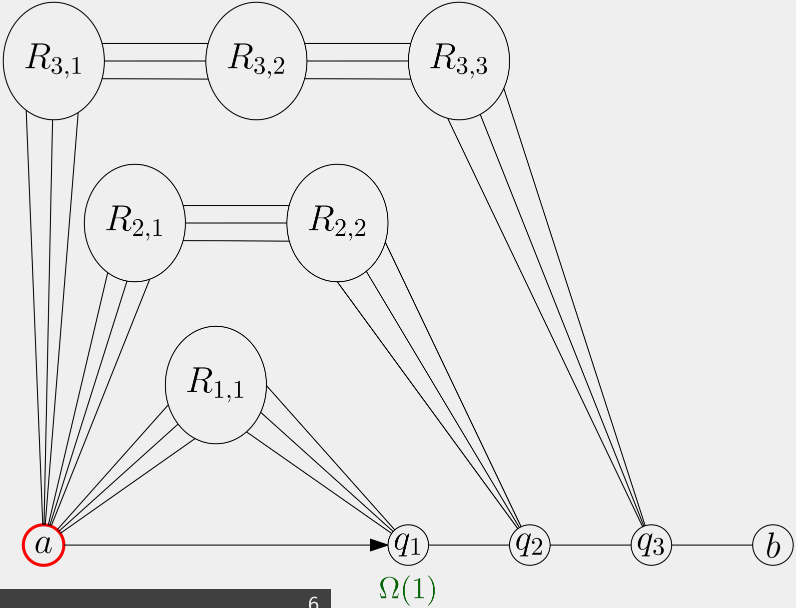
SUPERPOLYNOMIALITY - GENERAL IDEA



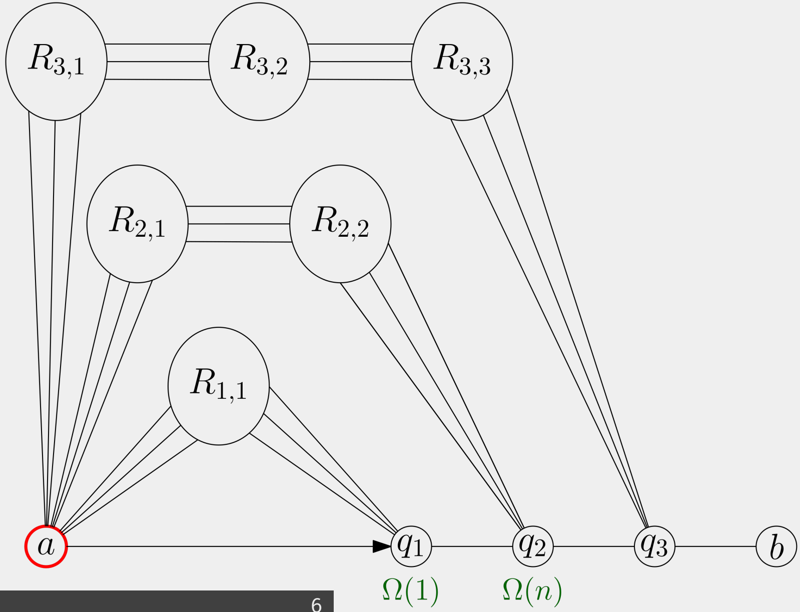


$R_{3,1}$
$R_{3,2}$
$R_{3,3}$
$R_{2,1}$
$R_{2,2}$
$R_{1,1}$
a
q_1
q_2
q_3
b

Superpolynomiality - general idea



Superpolynomiality - general idea



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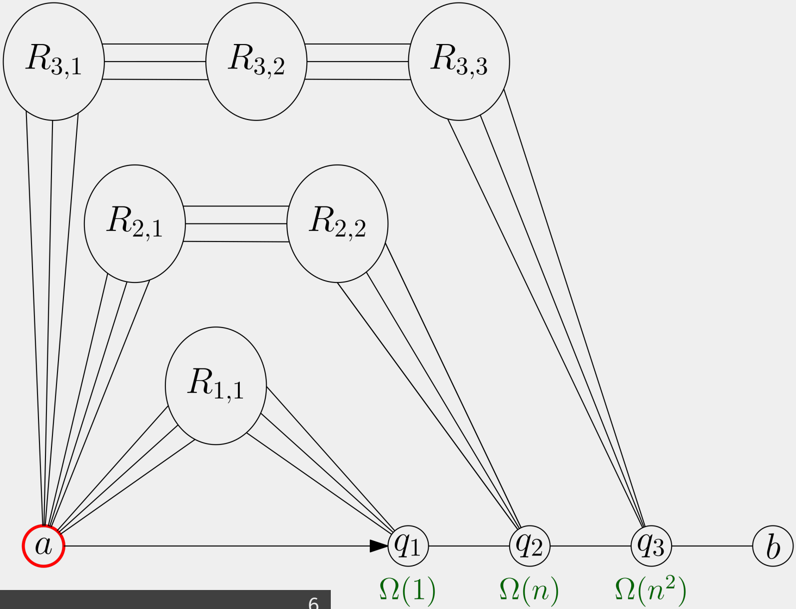

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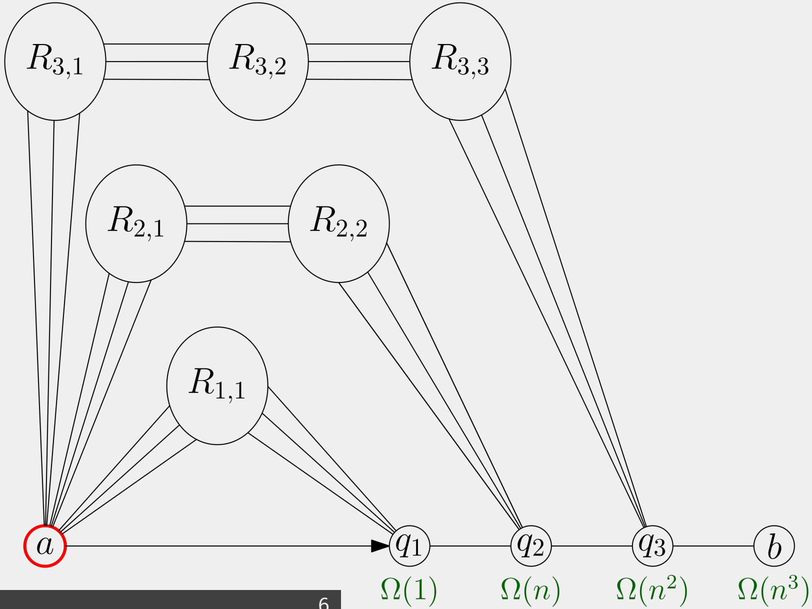

Upper bound

Lemma

The expected hitting time is at most $n \cdot (n-1)^{n-1}$.

Biased walk with bounded degree

Given $d \in \mathbb{N} : d \geq 3$ and a graph $G = (V, E)$ with $\max_{v \in V} \deg(v) \leq d$, choose some vertices $F \subseteq V$. In these 'excited' vertices, the random walker will deterministically take a step along a fixed shortest path.

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Question

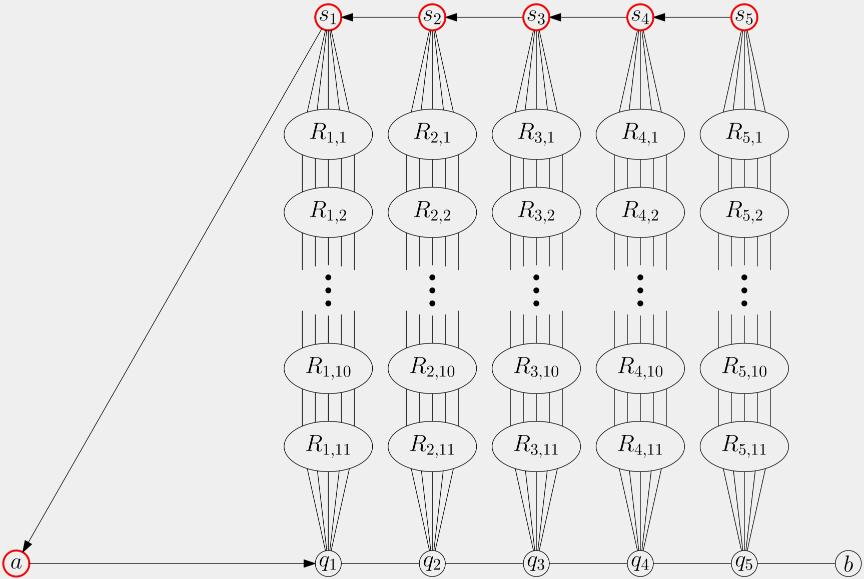
Is the expected hitting time still superpolynomial?

In the case of undirected connected graphs without excitation, the upper bound is $\mathcal{O}(n^2)$ [Aleliunas et al., 1979, F. Lawler, 1986].

We cannot use the identical approach as in the previous construction.

- We cannot make the probability of going ‘up’ arbitrarily high
- The vertex a must have its degree lowered

Bounded degree - new construction



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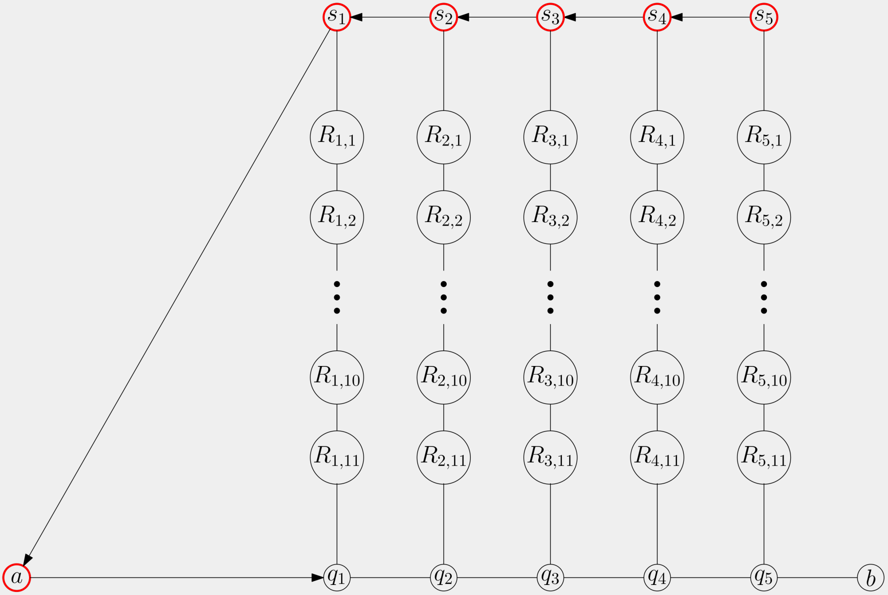


Bounded degree - limits

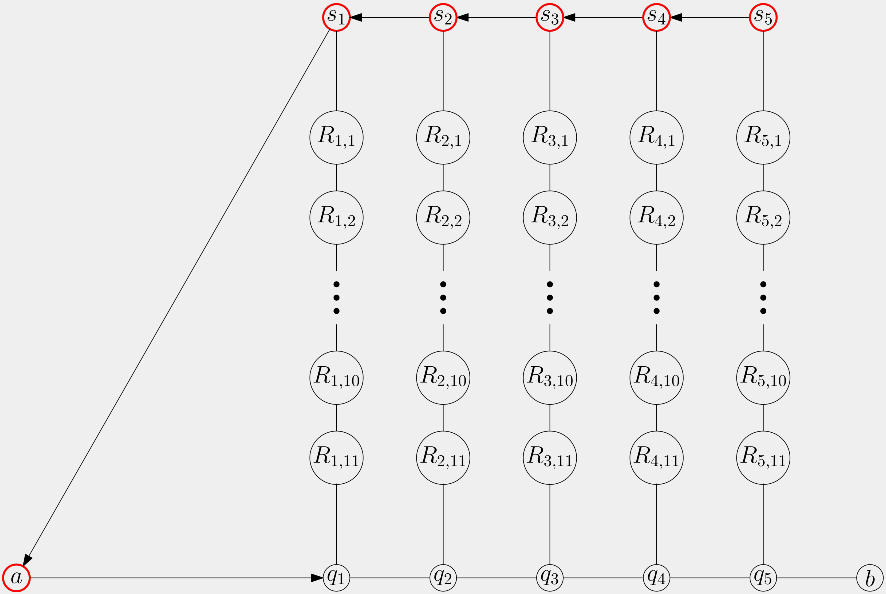
- More excited vertices necessary (roughly $\sqrt{n}$)

Bounded degree - upper bound

Lemma

The expected hitting time for maximum degree d is at most $n \cdot d^{n-1}$.

Bounded degree - proof idea



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