BIASED RANDOM WALKS

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Let $G = (V, E)$ be a graph, $a \neq b \in V$. A simple random walk is a randomly generated sequence of vertices $(v_i)$ such that $v_1 = a$, $v_{i+1} \in N(v_i)$ and $v_{i+1}$ is chosen uniformly at random.
Let $G = (V, E)$ be a graph, $a \neq b \in V$. A simple random walk is a randomly generated sequence of vertices $(v_i)$ such that $v_1 = a$, $v_{i+1} \in N(v_i)$ and $v_{i+1}$ is chosen uniformly at random.

The hitting time of $b$ is the number of steps the walk needs to reach $b$ from $a$. 


It was shown [Aleliunas et al., 1979, F. Lawler, 1986] that the expected hitting time on any connected undirected graph is of order $O(n^3)$. To be precise, the expected hitting time is at most $\frac{4}{27} n^3 - \frac{1}{9} n^2 + \frac{2}{3} n - 1$ [Brightwell and Winkler, 1990].
The main question

Biased walk

Given a graph $G = (V, E)$, choose some vertices $F \subseteq V$ and a target $b \in V$. In these ‘excited’ vertices, the random walker will deterministically take a step along a fixed shortest path to $b$. 
The main question

Biased walk

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**The Main Question**

**Biased walk**

Given a graph $G = (V, E)$, choose some vertices $F \subseteq V$ and a target $b \in V$. In these ‘excited’ vertices, the random walker will deterministically take a step along a fixed shortest path to $b$.

**Question**

Does the hitting time of $b$ change, and if so, how?
Can we show the same $O(n^3)$ bound on the expected hitting time as before?
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Problems and results

- Can we show the same $O(n^3)$ bound on the expected hitting time as before? No.
- Can we show any polynomial bound? No.
- Are there any other natural ‘biases’, which help the random walker?
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Are there any other natural ‘biases’, which help the random walker? ¯\_(ツ)_/¯
**Theorem**

For any \( c \in \mathbb{N} \), there exists a graph \( G = (V, E), |V| = n \) with vertices \( a, b \in V \) such that the expected hitting time of \( b \) when starting in \( a \) is \( \Omega(n^c) \). Moreover, only one excited vertex is required.
Superpolynomiality - General Idea

Diagram:

- Nodes labeled 'a' and 'b'
- 'Trap' node connecting to 'a' and 'b'

Graphical representation of 'Trap' relationships with 'a' and 'b'.
Superpolynomiality - General Idea

\[ R_{3,1} \quad R_{3,2} \quad R_{3,3} \]

\[ R_{2,1} \quad R_{2,2} \]

\[ R_{1,1} \]

\[ a \quad q_1 \quad q_2 \quad q_3 \quad b \]
Superpolynomiality - General Idea

Graph with nodes labeled $R_{3,1}$, $R_{3,2}$, $R_{3,3}$, $R_{2,1}$, $R_{2,2}$, and $R_{1,1}$. Edges connect these nodes, with a path from $a$ to $b$ through $q_1$, $q_2$, and $q_3$. The notation $\Omega(1)$ indicates a lower bound on a parameter or function.
SUPERPOLYNOMIALITY - GENERAL IDEA

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\[ R_{2,1} \quad R_{2,2} \]

\[ R_{1,1} \]

\[ a \quad q_1 \quad q_2 \quad q_3 \quad b \]

\[ \Omega(1) \quad \Omega(n) \]
SUPERPOLYNOMIALITY - GENERAL IDEA

\[ R_{3,1} \quad R_{3,2} \quad R_{3,3} \]

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\[ R_{1,1} \]

\[ a \rightarrow q_1 \rightarrow q_2 \rightarrow q_3 \rightarrow b \]

\[ \Omega(1) \quad \Omega(n) \quad \Omega(n^2) \]
Superpolynomiality - General Idea
Lemma

The expected hitting time is at most \( n \cdot (n - 1)^{n-1} \).
**Bounded degree**

Biased walk with bounded degree

Given $d \in \mathbb{N} : d \geq 3$ and a graph $G = (V, E)$ with $\max_{v \in V} \deg(v) \leq d$, choose some vertices $F \subseteq V$. In these ‘excited’ vertices, the random walker will deterministically take a step along a fixed shortest path.
Biased walk with bounded degree

Given \( d \in \mathbb{N} : d \geq 3 \) and a graph \( G = (V, E) \) with \( \max_{v \in V} \deg(v) \leq d \), choose some vertices \( F \subseteq V \). In these ‘excited’ vertices, the random walker will deterministically take a step along a fixed shortest path.

Question

Is the expected hitting time still superpolynomial?

In the case of undirected connected graphs without excitation, the upper bound is \( O(n^2) \) [Aleliunas et al., 1979, F. Lawler, 1986].
We cannot use the identical approach as in the previous construction.

- We cannot make the probability of going ‘up’ arbitrarily high.
- The vertex $a$ must have its degree lowered.
BOUNDDED DEGREE - NEW CONSTRUCTION
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\[ a \rightarrow q_1 \rightarrow q_2 \rightarrow q_3 \rightarrow q_4 \rightarrow q_5 \rightarrow b \]

Nodes: \( a, q_1, q_2, q_3, q_4, q_5, b \)

Edges: "bounded degree" connections between nodes.

Labels: \( R_{i,j} \) for each connection.
More excited vertices necessary (roughly $\sqrt{n}$)
Lemma

The expected hitting time for maximum degree $d$ is at most $n \cdot d^{n-1}$.
Bounded Degree - Proof Idea

Diagram showing nodes and edges labeled with indices and variables.
REFERENCES

