Poznámky - Grafové minory a stromové rozklady

Petr Chmel, ZS 2020/21

Definition 1 (Minor, induced, topological minor)**.** A graph H is

- a *minor* of a graph G if it can be obtained from a subgraph of G by a series of contractions
- an *induced minor* of a graph G if it can be obtained from an induced subgraph of G by a series of contractions
- *topological minor* of a graph G if it can be obtained from a subgraph of G by a series of topological contractions (the inverse operation to a subdivision),

Lemma 1 (Minors and models)**.** A graph H is a minor of $G \Leftrightarrow$ there exists a map $f : V(H) \to \mathcal{P}(V(G))$ satisfying the following:

1. $\forall u \in V(H) : f(u)$ is nonempty and connected
2. $\forall u, v \in V(H) : f(u) \cap f(v) = \emptyset$
3. $\{u, v\} \in E(H) \Rightarrow \exists \{u', v'\} \in E(G) : u' \in f(u), v' \in f(v)$.

If talking about induced minors, then 3) is an equivalence.

Proof. "$\Rightarrow$": Every $u \in V(H)$ is obtained by a series of contractions of a connected subgraph: the vertices of the subgraph are the result of the map (disconnected components cannot combine), which implies 1. Property 2 is obvious by definition. Property 3 is proven by contrapositive: if $\{u', v'\}$ does not exist for any $u' \in f(u), v' \in f(v)$, then there cannot be an edge $\{u, v\}$.

"$\Leftarrow$": We contract $f(u)$ into u, the conditions guarantee it works (TODO). ⊞

Lemma 2 (Topological minors and minors with small degree)**.** If H has $\Delta(H) \leq 3$, then H is a minor of G $\Leftrightarrow$ H is a topological minor of G.

Proof. "$\Leftarrow$": Simple: every topological minor is a "normal" minor.

"$\Rightarrow$": H is a minor, and hence there exists a map $f : V(H) \to \mathcal{P}(V(G))$ satisfying the following:

1. $\forall u \in V(H) : f(u)$ is nonempty and connected
2. $\forall u, v \in V(H) : f(u) \cap f(v) = \emptyset$
3. $\{u, v\} \in E(H) \Rightarrow \exists \{u', v'\} \in E(G) : u' \in f(u), v' \in f(v)$.

We first connect each adjacent $f(u)$, $f(v)$ by a single edge. Next, we find a spanning subgraph connecting the edge ends in the respective subsets. Finally, we do topological contractions, which can be done as H is of maximum degree three and hence the spanning subgraphs are either paths, single vertices or subdivided three-star ($K_{1,3}$). ⊞

Observation (Transitivity of being a minor)**.** $F \leq_m H, H \leq_m G \Rightarrow F \leq_m G, G \leq_m H \wedge H \leq_m G \Rightarrow G \cong H$.

Definition 2 (Closed classes)**.** A class $\mathcal{G}$ is closed on taking minors if $G \in \mathcal{G} \wedge H \leq_m G \Rightarrow H \in \mathcal{G}$.

Observation (Observations on closed classes)**.** If $\mathcal{G}$ is closed on minors, then it is also closed on induced minors. Also, if $\mathcal{G}$ is not closed on induced minors, then it is also not closed on general minosr.

Planar graphs are closed on minors.

Theorem 1 (Kuratowski-Wagner)**.** The following are equivalent:

- G is planar
- G has no subdivision of $K_{3,3}$ or K_5 as a subgraph

- G has neither $K_{3,3}$ nor K_5 as a minor

Proof. 3⇒ 2 is trivial: if G has either subdivision, it has a topological minor, which is a minor.

2⇒3: if graph has a $K_{3,3}$ as a minor, then it has a $K_{3,3}$ as a topological minor, and hence it has a subdivision of $K_{3,3}$. If the graph has a K_5 minor, then either it has a K_5 as a topological minor, or we contracted an edge of high degree. Then, we can find a minor of $K_{3,3}$. ⊞

Definition 3 (Outerplanar graph)**.** A graph is outerplanar if it allows a drawing where every vertex is nicident with the outer face.

Theorem 2 (Forbidden minors of outerplanar graphs)**.** G is outerplanar, if and only if it has no K_4 nor $K_{2,3}$ minor.

Proof. "⇐" By contrapositive: if it has one of the minors, they are obviously the obstructions.

"⇒" If it has neither K_4 nor $K_{2,3}$, then it is surely planar, hence we can assume it is planar but not outerplanar. Therefore in every planar drawing, there exists a vertex inside the complement of the outerface. It must have a degree at least two (otherwise we could just draw it outside) and either the two neighbors must be nonadjacent (we could put it outside otherwise) or it has at least three neighbors which form a triangle. In the first case, we get a $K_{2,3}$, in the second, we get K_4.

Other possibilities: adding some random vertex, duals of outerplanar graphs are simple forests, or induction and ear lemma. ⊞

Definition 4 (Quasiordering)**.** A quasiordering is a transitive, reflexive relation.

Definition 5 (Well-ordered class)**.** A well-ordered class is $(X, \leq)$ such that for every infinite sequence $(a_i)_{i=1}^{\infty} \subset X$, there exist $a_i, a_j : ai \leq a_j \wedge i < j$. (There is no infinite decreasing sequence.)

Observation. In a well-ordering, every sequence has a nondecreasing subsequence.

Proof. Using infinite Ramsey theorem: one color for good pairs, other for bad pairs. ⊞

Definition 6 (Set of forbidden elements)**.** For $(X, \leq)$ a quasiordering, $Y \subseteq X$, we say that O is a set of forbidden elements for Y, if $\forall x \in X : x \notin Y \Leftrightarrow \exists o \in O : o \leq x$.

Observation. O exists if and only if Y is a down-set (dolní množina).

Furthermore, if Y is closed downwards, then O exists and is finite as the set of minimal elements of $X \setminus Y$ (factored by the equivalence classes).

Theorem 3 (Higman)**.** Let $(X, \leq)$ be a well-ordering. Then $(X^{\text{fin}}, \leq)$ is well-ordering, where X^{fin} is a set of finite subsetets of X, where $A \leq B$ iff there exists an injection $f : A \to B$ such that $\forall a \in A : a \leq f(a)$.

Proof. Proof by contradiction: let $(X, \leq)$ we a well ordering, but let there exist $(A_i)_{i=1}^{\infty}$ such that $\forall i : A_i$ is chosen as the smallest such that it permits a non-well extension. Then, we set $a_i \in A_i$ arbitrarily, $B_I = A_i \setminus \{a_i\}$. We know that $(X, \leq)$ is a well-ordering, and hence a_i has a non-decreasing subsequence $a_{i_1}, \ldots$. By the choice of A_i, the sequence $A_1, \ldots, A_{i_1-1}, B_{i_1}, B_{i_2}, \ldots$ must have a good pair, which yields a good pair in (A_i), which gives us a contradiction. ⊞

Theorem 4 (Kruskal)**.** The class of finite rooted trees is well quasiordered by the topological minor relation.

Proof. Proof by contradiction: let's find $(T_i)_{i=1}^{\infty}$ such that $\forall i$, T_i is the smallest tree permitting a non-well extension. We denote r_i as the root of T_i, $F_i := T_i \setminus \{r_i\}, \mathcal{F} := \bigcup_i F_i$.

First, we show that $\mathcal{F}$ is well-ordered. Taking an arbitrary sequence $(B_j)_{j=1}^{\infty}$, $B_i \in \mathcal{F}$ and for each j, we set $i(j)$ such that B_j is a component of $F_{i(j)}$ and we take the subsequence $B_{j_1}, \ldots$ such that $i(j_1) \leq i(j_2) \ldots$ byla neklesající. Therefore, $T_1, \ldots, T_{i(j_1)-1}, B_{j_1}, B_{j_2}, \ldots$ is well, and therefore $\mathcal{F}$ is well-ordered.

Now, every $F_i \in \mathcal{F}^{\text{fin}}$, therefore $(\mathcal{F}^{\text{fin}}, \leq)$ is well-ordered (by Higman theorem), and therefore the sequence $(F_i)_{i=1}^{\infty}$ has a good pair (F_i, F_j), therefore (T_i, T_j) is a good pair as well. ⊞

Definition 7 (Tree decomposition)**.** A tree decomposition of a graph Gsi a tree T such that the vertices of T (nodes) are subsets of $V(G)$ and

1. $\forall e = (u, v) \in E(G) : \exists X_i \in V(T) : \{u, v\} \subseteq X_i$

2. $\forall u \in V(G)$: the nodes of T containing u form a nonempty connected subgraph (a subtree in particular)

Definition 8 (Treewidth)**.** The width of a tree decomposition T is $\max_{X_i \in V(T)} |X_i - 1|$.
The treewidth $\text{tw}(G)$ is the minimum possible width of any tree decomposition of G.

Observation (On connectivity condition)**.** The connectivity condition in the definition of the tree decomposition can be equivalently expressed as follows: "if X_k is between X_i and X_j on T, then $X_i \cap X_j \subseteq X_k$"

Observation (Treewidth of a disconnected graph)**.** The treewidth of a disconnected graph is the maximum of treewidths of its connected components.

Lemma 3. $\text{tw}(G) \geq \omega(G) - 1$

Proof. We show that for any clique K of graph G, there is a node $X \in V(T)$ such that $K \subseteq X$.
By contradiction: let there be T, K such that $\forall X \in V(T) : K \setminus X \neq \emptyset$. We direct the edges of T towards the missing vertices of the set. There has to exist an edge $\{X_i, X_j\}$ which is directed both ways as every bag directs at least one edge. Therefore we get that for $u \in K \setminus X_i, v \in K \setminus X_j, u \neq v$ there is no bag containing the edge $\{u, v\}$ of the clique, a contradiction. ⊞

Theorem 5 (Treewidth and minors)**.** If H is a minor of G, then $\text{tw}(H) \leq \text{tw}(G)$.

Proof. There are three operations which we apply on the graph and change the tree decomposition accordingly:

- vertex removal: just remove the vertex from the bags
- edge removal: we don't have to do anything
- edge contraction: rename one vertex to the other, as they shared an edge, the subtree induced by the vertex remains connected

⊞

Remark (Treewidths of some classes/graphs)**.**

- $\text{tw}(T)$ for T any tree is 1
- $\text{tw}(Q_3) = 3$
- $\text{tw}(K_{m,n}) = \min(m, n)$
- $\text{tw}(P) \leq 2$ for any P outerplanar graph
- $\text{tw}(G) = 1 \Rightarrow G$ is a forest

Definition 9 (k-terminal graph, gluing)**.** A k-terminal graph is a graph where k distinct vertices are marked as terminals $t_1, \ldots, t_k$.
We say that $\odot$ is an n-ary *gluing operation of k-terminal graphs* if for $H = \odot(G_1, \ldots, G_n)$ if thods that H was obtained from the disjoint union of $G_1, \ldots, G_m$ by merging presribed sets of terminal – in each merge, at most one terminal from each G_i perticipates. Moreover, the terminals of H are also prescribed with terminals of $G_1, \ldots, G_n$ or the merged vertices.

Definition 10 (Recursively defined classes)**.** A recursively defined graph class of k-terminal graphs is given by a pair (O, B), where O is a set of operations and B is a baseset of base graphs.
We say that a graph G belongs to the class if it can be obtained from the base graphs by the applications of operations. (This can be denoted by a composing tree.)

Example 1 (Binary operations on 2-terminal graphs)**.** $\odot_s$ - serial join: joins t_2 of G_1 with t_1 of G_2, and t_i of G_i is the t_i of the result
$\odot_p$ - parallel join: joins t_1's together and t_2's together, the resulting vertex is the t_i
$\odot_j$ - jackknife join: joins t_2 of G_1 with t_1 of G_2, and t_i of G_1 is the t_i of the result

Observation (A simple recursively defined class)**.** The class given by $(\{\odot_j\}, \{K_2\})$ is precisely the class of trees.

Definition 11 (Series-parallel graphs)**.** Series-parallel graphs are precisely the graphs given by $(\{\odot_s, \odot_p\}, \{K_2\})$.

Theorem 6 (Series-parallel graphs have treewidth ≤ 2)**.** Series parallel graphs and their subgraphs are precisely the graphs of treewidth ≤ 2.

Proof. Two goals: if series-parallel graph, then treewidth is at most two, and if treewidth is at most 2, then show it is a subgraph of series-parallel graphs.

First, given a series-parallel graph, we show the treewidth is at most two. The base graph, K_2 has $\text{tw}(K_2) = 1 \leq 2$. Overall, we seek rooted decompositions which have precisely the two terminal vertices in the root.

Series operation: we join them with a node with the three vertices and then create a new root with just the two noew terminals.

Parallel operation: just glue the tree together with the root intact.

The other direction, given a graph G with $\text{tw}(G) \leq 2$, we show that it is a subgraph of some series-parallel graph. Without loss of generality, we may assume that every node of T induces a K_3 (if not, add an edge and it does not break the treewidth, or add a vertex from one of the neighbors). For $\{X_i, X_j\} \in E(T)$, we get that $|X_i \cap X_j| = 2$ (otherwise we may add more intermediate bags).

For any leaf, we get that the bag consists of three nodes u, v, w, then one of the three nodes does not appear anywhere else in the tree decomposition. By induction hypothesis and the construction tree for G'', some leaf of the tree corresponds to $K_2 = \{u, v\}$. ⊞

Definition 12 (Chordal graphs)**.** A graph is chordal, if it has no hole (hole is an induced cycle of length ≥ 4).

Proposition 1 (Simplicial vertices of chordal graphs)**.** Every chordal graph has a simplicial vertex.

Proof. By contradiction: let there for every vertex $v \in V$ be two neighbors $x \neq y \in N(v)$ with $\{x, y\} \notin E$. Then a minimal cut (denoted by S_v) which divides x and y must be a clique (we would get a chordless cycle otherwise). Then we denote $A_1, \ldots, A_k$ the components of $G \setminus S_v$. Any other S_u lies inside $A_i \cup S_v$ for some i. Therefore the cuts do not cross (in some sense). Let us take u such that one of the components $G \setminus S_u$ is the smallest possible and denote it by A. Then for any $w \in A$, it hodls that S_w divides a even smaller component inside $A \cup S_v$, a contradiction. ⊞

Corollary 1. Every chordal graph has a tree decomposition where every node induces a clique.

Proof. G has a simplicial vertex u. $G - u$ has such decomposition by induction hypothesis, we just add the new vertex either to the remaining clique, or we add a new node. ⊞

Definition 13 (k-tree)**.** A k-tree is either a K_{k+1}, or a graph which is created from a k-tree by adding a simplicial vertex to a clique of size k.

Proposition 2 (Treewidth and k-trees)**.** A graph G has treewidth at most k if and only if it is a subgraph of a k-tree

Proof. "$\Leftarrow$": By definition, the treewidth of k-tree is k, hence any subgraph has treewidth at most k.

"$\Rightarrow$": Given a tree decomposition T, we extend the decomposition by adding nodes, edges and vertices so that the tree-decomposition corresponds to a k-tree. ⊞

Corollary 2. $\text{tw}(G) := \min\{\omega(G') - 1 : G \subseteq G', G' \text{ is chordal}\}$

Definition 14 (Separator)**.** By a separator of A and B we mean a partition of $V(G)$ into three parts A, B, S such that every every path from $a \in A$ to $b \in B$ goes through S.

Proposition 3. If X_i, X_j is an edge of a tree decomposition T of a graph G, then (at least) one of the following holds:

- the intersection $X_i \cap X_j$ is a separator in G,

- all nodes of the component $T - \{\{X_i, X_j\}\}$ containing the node X_i are subsets of X_j,
- all nodes of the component $T - \{\{X_i, X_j\}\}$ containing the node X_j are subsets of X_i.

Proof. By contradiction: assume that neither of the last two holds. Then there has to exist a path outside of $X_i \cap X_j$, however the path must cross the edge, and hence there is one vertex in both X_i and X_j. ⊞

Observation. For every tree decomposition T and each $W \subset V(G)$, it holds that either $W \subseteq X_i$ for $X_i \in V_t$ or there exists an edge $(X_i, X_j) \in E(T)$ such that W reaches in both parts ot $G \setminus (X_i \cap X_j)$.

Proof. Either the first case from the previous proposition holds, or $G[W]$ does not induce a clique, hence there are a, b not connected by an edge in G such that $a \in X_i \setminus X_j, b \in X_j \setminus X_i$. ⊞

Definition 15 (Good separator)**.** We say that S is a good separator for $W \subseteq V(G)$ if both A and B have at most 2/3 vertices of W ($|A \cap W| \leq \frac{2}{3}|W|$).

Observation (Treewidth and good separators)**.** If $\mathrm{tw}(G) \leq k$, then each W has a good separator of size at most $k+1$.

Proof. Let T be a tree decomposition with all nodes of size $\leq k+1$. Then we may assume without loss of generality that each node has degree ≤ 3 (we split the vertex into two otherwise) and then, there can be at most one subtree with more than 2/3 vertices of W. However, then the other two subtrees together combine for less and a contradiction quickly follows. ⊞

Theorem 7 (Reed)**.** If every $W \subseteq V(G)$ has a good separator of size $\leq k+1$, then $\mathrm{tw}(G) \leq 4k+3$.

Algorithm 1 (Tree decomposition - Reed)**.** Input: a graph G, $W \subseteq V(G) : |W| = 3k+3$, output: tree decomposition T of G with widrth $\leq 4k+3$, where $W \subseteq X_{\mathrm{root}}$. Runs in time $\mathcal{O}(27^k \cdot n^2)$.

```
TreeDecomp(G,W):

if |V(G)|\leq 4k+4, put all in one bag and return

otherwise find a good separator S for W in size at most k+1 as follows:
    try all partitions of W into W_A, W_B, W_S and expand W_S into S
    (possible if there are no k+2-|W_S| disjoint paths between W_A and W_B)
    If we fail for all possible cases, then W has no separator and we fail

if this fails, then halt, as treewidth is more than k
otherwise S separates A and B
Then G_A = G[A\cup S], W_A=S\cup (W\cap A), T_A = TreeDecomp(G_A, W_A)
Also G_B = G[B\cup S], W_B=S\cup (W\cap B), T_B = TreeDecomp(G_B, W_B)
(Note that W_A, W_B might need to add some vertices)

Set T = T_A\cup T_B \cup X_{root}, where X_root = W\cup S and return T
```

Complexity: 3-partitions in $\mathcal{O}(3^{|W|}) = \mathcal{O}(3^{3k+3}) = \mathcal{O}(27^k)$, and the extension can be found via flows in $\mathcal{O}(k \cdot |E(G)|) = \mathcal{O}(k^2 n)$ time as there are at most kn edges in the graph.

Definition 16 (Strongly linked set)**.** A set $W \subseteq V(G)$ is called *strongly k-linked* if it has no good separator with at most k vertices.

Theorem 8 (Alan, Seymour, Thomas: planar graphs and grid minors)**.** Every planar graph of treewidth $\geq 20k - 12$ contains a $k \times k$ grid as a minor.

Proof. $\mathrm{tw}(G) \geq 20k - 12$, therefore G contains stronly $(5k-3)$-linked set W.We take the planar drawing if G and identify it with a closed Jordan curve φ such that

- φ intersects the drawing only in vertices

- φ contains at most K vertices
- the interior of φ contains at least 2/3 vertices of W
- the interior of φ contains as few vertices as possible

Then without loss of generality, we may assume that φ contains exactly $4k$ vertices (the curve, not the inside). If not, then we can reroute it slightly, Then, we denote the vertices $v_1, \dots, v_{4k}$.

If there are no k disjoint paths $c_1, \dots, c_k$ where c_i is path from v_i to v_{2k+i}, there is a cut of size $\leq k-1$ separating $v_1, \dots, v_k$ from $v_{2k+1}, \dots, v_{3k}$ and therefore we could get a φ with a smaller number of vertices in the interior: there exists a curve ψ through the cut. Due to the choice of φ: there are $\varphi_1 \cup \psi$ separating at most 2/3 vertices of W, and the same holding for $\varphi_2 \cup \psi$, we get that $\varphi \cup \psi$ shows that W is not strongly $(5k-3)$-linked. Therefore the column and row paths exist, and therefore we have the required minor. ⊞

Definition 17 (Bramble, its order)**.** A bramble in a graph G is a set system $\mathcal{B} = \{B_1, \dots, B_n\} \subseteq \mathcal{P}(V_G)$ such that

- each $b \in \mathcal{B}$ induces a connected subgraph
- for each pair of sets in the bramble, their union induces a connected subgraph

The order of a bramble $\mathcal{B}$ is the size of the smallewst W that intersects all $B \in \mathcal{B}$.

Theorem 9 (A single direction of strong duality)**.** *If G has a bramble of order $k+1$, then $\mathrm{tw}(G) \geq k$.*

Proof. We show that in any decomposition T, some node X intersects all $B \in \mathcal{B}$. By contradiction if not: we direct the edges of T incident with X towards a set from $\mathcal{B}$ disjoint with X. Then, there is a bidirected edge, which implies that there exist two separated sets in the bramble, a contradiction. ⊞

Theorem 10 (Brambles and tree decomposition)**.** *If G has no bramble of order $k+1$ then for every bramble $\mathcal{B}$ there is a tree decomposition such that $\forall$ node of size $\geq k+1$ is a leaf in T and does not cover $\mathcal{B}$.*

Proof. TODO ⊞

Theorem 11 (Minimax theorem, Seymour, Thomas)**.** $\max_{\mathcal{B} \text{ bramble}} \min_{W \text{ convers } \mathcal{B}} |W| = \min_{T \text{ tree decomp.}} \max_{X \text{ bag}} |X|$.

Proof. Prove by contrapositive: if G heas no bramble of order $k+1$, then the treewidth is $\leq k$. Then we apply the previous theorem for $\mathcal{B} = \{V_G\}$ to get T - there, all nodes must be of size $\leq k$ as each nontrivially covers $\mathcal{B}$, and hence $\mathrm{tw}(G) < k$. ⊞

Definition 18 (Nice tree decomposition)**.** A tree decomposition T is nice, if T is rooted, every node X_i is either

- a leaf
- it has two children X_j, X_k and $X_i = X_j = X_k$ (*join node*)
- it has a single child X_j and either $X_i = X_j \cup \{u\}$ (*introduce node* for u) or $X_j = X_i \cup \{u\}$ (*forget node* for u).

Theorem 12 (Optimal nice tree decompositions)**.** *Every graph G has a nice tree decomposition of width $\mathrm{tw}(G)$.*

Proof. Given an optimal T', we root the tree, replace leaves with paths, fix joins and fix paths between differing nodes. ⊞

Theorem 13. *For every k there is a graph G on $3k$ vertices such that any of its nice tree decompositions with one-vertex leaves has size $\Omega(k^2)$.*

Proof. We take a pairing on $2k$ vertices u_i, v_i and then add a single vertex w_1 joined to u_i. Then we add $w_2, \dots, w_k$ as universal vertices. Then $\mathrm{tw}(G) = k, X_i = \{v_i, u_i, w_2, \dots, w_n\}$. Then there are paths of length k, there are k of them and hence there are at least $\Omega(k^2)$ nodes. ⊞

Theorem 14. For every G, there is an optimal nice T with at moste $4|V_G|$ nodes (if we do not require single-vertex leaves).

Proof. WLOG G is a k-tree - we choose u simplicial in G. By induction, $G' = G - u$ has a T' with $\leq 4|V(G')|$ nodes. Then $T' \to T$ is allowed to use at most four nodes. If the node is a leaf, we need two nodes: $X_i \to N(u) \to N[u]$. If the node is not a leaf, we need a split (two nodes) and two nodes from the previous (we now have a leaf). ⊞

Definition 19 (Path decomposition, pathwidth)**.** Path decomposition is a tree decomposition, where T is a path.

Pathwidth is the minimum possible width of a path decomposition. (Note that $\text{tw}(G) \leq \text{pw}(G)$.)

Theorem 15. For T_k, a rooted ternary tree with $k+1$ levels, $\text{pw}(T_k) \geq k$.

Proof. By induction: for $n = 1$, easy.

Let P be a path decomposition of T_k. Then we can join the decompositions, however there has to be a single vertex that is staying from the leftmost to the rightmost or vice versa. Therefore $\text{pw}(T_{k+1}) \geq k+1$. ⊞

Theorem 16 (Pathwidth and treewidth)**.** For any G on n vertices: $\text{pw}(G) \in \mathcal{O}(\text{tw}(G) \log n)$.

Proof. We start with an optimal tree decomposition T with $\leq n$ nodes. The we define labels $L : V_T \to \mathbb{N}$ recursively: $L(X_{leaf}) = 1$, $L(X_i) = \max(L(X_j), L(X_k) + 1)$ where $L(X_j) \geq L(X_k) \geq \ldots$.

We can just build the decomposition incrementally and the width corresponds to the width of $T \cdot \log n$ as it is at most $\text{tw}(G) \cdot L(X_{\text{root}})$, and the label is at most logarithmic. ⊞

Definition 20 (Branch decomposition, branchwidth)**.** Branch decomposition is a tree B with degrees 1 or 3, with a bijection between the leaves of B and *edges* of G. Any edge f of B divides $E(G)$ into two partitions $A_f, \overline{A_f}$. The width of an edge $f \in E(B)$ is the number of vertices both in A_f and $\overline{A_f}$. The width of the decomposition is the maximum width of an edge.

The branchwidth of a graph G is the minimum width of a branch decomposition of G.

Proposition 4 (Minors and branchwidth)**.** If H is a minor of G, then $\text{bw}(H) \leq \text{bw}(G)$.

Proof. Removal of an edge: we just remove a leaf (and supress the vertex of degree 2).

Removal of a vertex: remove the leaves with edges and then supress the vertices of degree 2.

Contractions: remove the edge $\{u, v\}$ and then rename the vertices. ⊞

Proposition 5 (Branchwidth and treewidth)**.** If $\text{bw}(G) \geq 2$, then $\text{bw}(G) \leq \text{tw}(G) + 1 \leq \lfloor \frac{3}{2}\text{bw}(G) \rfloor$.

Proof. Given a tree decomposition, we do the following:

- For every edge, we take some node $X_i \ni r$ and we add e as a leaf
- We do similar changes on other vertices as ib trees
- The weights of original edges are given by $X_i \cap X_j$
- The weights of new edges are bounded by $|X_i|$

Hence $\text{bw}(G) \leq \text{tw}(G) + 1$.

Next, given a branch decomposition, we take it as a tree decomposition. The size of X is surely less than the sum of widths of edges divided by two: the vertices in the nodes are the ones which add one to widths, and since the vertices are internal, they have degree three and hence the width is at most $\lfloor \frac{3}{2}\text{bw}(G) \rfloor$. ⊞

Definition 21 (NLC-width)**.** A build-tree for G is a tree with G as the root node and three types of nodes:

- a leaf: creation of a vertex v with a label $i(v)$
- Q_r with $r : I \to I$ - a relabeling of the son by r
- $\oplus_s$, $s \subseteq I \times I$ - two children, creates edges between label i in the left son and j in right son $\forall (i, j) \in S$.

NLC-width (node-labelled controlled width) is the least necessary number of labels needed to build G.

Theorem 17 (NLC-width and treewidth)**.** $\mathrm{nlcw}(G) \leq 2^{\mathrm{tw}(G)+1} + \mathrm{tw}(G) + 1$

Proof. TODO ⊞

Definition 22 (Rank decomposition, rankwidth)**.** Rank decomposition is a ternary tree R (unrooted, every internal vertex has degree 3) with leaves in one-to-one correspondence with the vertices of the given graph.

For an edge e, the matrix A_e is a $\{0, 1\}$-matrix with rows indexed by one component of the decomposition $R - e$ and columns indexed by the other with 1 being iff the two vertices share an edge.

The weight of an edge e is $\mathrm{rank}(A_e)$. The width of R is the maximum weight of an edge. The rankwidth is the minimum possible width over all rank decompositions.

Theorem 18. $\mathrm{rw}(G) \leq \mathrm{nlcw}(G) \leq 2^{\mathrm{rw}(G)}$

Proof. TODO ⊞

Definition 23 (Local complement)**.** A local complement is an operation $G \to G * v$ obtained by eschange of the subgraph $G[N(v)]$ by $\overline{G[N(v)]}$ in G.

Definition 24 (A vertex minor)**.** A vertex minor H of G is obtained from an induced subgraph of G by local complements.

Proposition 6. Local complements do not alter rankwidth.

Proof. TODO ⊞

Corollary 3. if H is a vertex minor of G, then $\mathrm{rw}(H) \leq \mathrm{rw}(G)$.

Theorem 19 (Oum, '07)**.** Graphs of bounded rankwidth are well-quasi-ordered by vertex minors.

List of topics